JOURNAL OF APPROXIMATION THEORY **50**, 297–325 (1987)

Interpolation Theorems and Expansions with Respect to Fourier Type Systems

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Communicated by V. Totik

Received September 26, 1984

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0. INTRODUCTION

0.1. In the first named author's investigations summarized in his monograph [1], on the basis of the remarkable asymptotic properties of the entire functions of Mittag–Leffler type and of order ρ,

$$E_\rho(z;\mu)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+k/\rho)}\qquad(\mu>0,\ \rho>0),\tag{0.1}$$

a complete theory of harmonic analysis was constructed for an arbitrary system of rays with origin at the point $z=0$ of the complex plane. These results may be considered as highly developed analogs of the classical Plancherel theorem on the Fourier transform in L_2.

It is a well-known fact that every function $\varphi(x)\in L_2(0,\sigma)$ $(0<\sigma<+\infty)$ may be expanded in any of the two Fourier series with respect to the system of trigonometric functions

$$\left\{\sqrt{\frac{2}{\sigma}}\sin\frac{\pi k}{\sigma}x\right\}_1^{\infty}\qquad\text{and/or}\qquad\left\{\frac{1}{\sqrt{\sigma}},\left\{\sqrt{\frac{2}{\sigma}}\cos\frac{\pi k}{\sigma}x\right\}_1^{\infty}\right\}.\tag{0.2}$$

0021-9045/87 $3.00

The Plancherel theorems on cos and sin Fourier transforms in $L_2(0, +\infty)$ are essentially continuous analogs of Fourier series with respect to the systems (0.2) for $\sigma = +\infty$.

Among the general results of [1] there is a particular proposition which contains the above mentioned Plancherel theorems on cos and sin Fourier transforms in $L_2(0, +\infty)$. Below we shall formulate that result and two theorems from [1] which play a crucial role in the present investigation.

(a) First, we introduce the following notations. Let $L_2^{\mu}(0, +\infty)$ be the set of measurable functions g on $(0, +\infty)$ such that

$$\int_0^{+\infty} |g(y)|^2 y^{2(\mu-1)} \, dy < +\infty, \qquad (\tfrac{1}{2} < \mu < \tfrac{5}{2}). \tag{0.3}$$

If, for $g(y) \in L_2^{\mu}(0, +\infty)$ and for $\{g_a(y)\} \subset L_2^{\mu}(0, +\infty)$ $(0 < a < +\infty)$, we have

$$\lim_{a \to +\infty} \int_0^{+\infty} |g(y) - g_a(y)|^2 y^{2(\mu-1)} dy = 0 \tag{0.4}$$

then, for brevity, we write

$$g(y) = \underset{a \to +\infty}{\overset{(\mu)}{\text{l.i.m.}}} \; g_a(y). \tag{0.4'}$$

THEOREM 1.[1] *Let* $\mu \in (\frac{1}{2}, \frac{5}{2})$ *and* $g(y) \in L_2^{\mu}(0, +\infty)$. *Then, in the* $L_2(0, +\infty)$ *metric,*

$$f(x) = \underset{a \to +\infty}{\text{l.i.m.}} \frac{2}{\pi} \int_0^a \cos\left[xy + \frac{\pi}{2}(1-\mu)\right] g(y)\, y^{\mu-1}\, dy \tag{0.5}$$

and the inversion formula

$$g(y) = \underset{a \to +\infty}{\overset{(\mu)}{\text{l.i.m.}}} \int_0^a E_{1/2}(-y^2x^2; \mu)\, x^{\mu-1} f(x)\, dx \tag{0.6}$$

holds.

It is easy to see that the Plancherel theorems on cos and sin Fourier transforms follow from this theorem for $\mu = 1$ and $\mu = 2$.

One of the main purposes of this paper is to find discrete analogs of the formulae (0.5) – (0.6) of Theorem 1. In other words, we are interested in series expansion theorems similar to Fourier expansions with respect to the systems (0.2).

[1] See [1, Theorem 4.2] for $\rho = \frac{1}{2}$.

(b) In [1] some rather general theorems were also obtained on a parametric representation of an entire function of finite order $\rho \geqslant \frac{1}{2}$ and of normal type $\leqslant \sigma$ such that the square of its module is integrable with respect to the weight r^{ω} $(-1<\omega<1)$ along a given system of rays with origin at the point $z=0$ (see [1, Theorems 6.10–6.16]).

Now we state the second theorem which we use below and which is a simple generalization of the well-known Paley–Wiener result.

THEOREM II.[2] *The class* $W^{2,\omega}_{1/2,\sigma}$ $(-1<\omega<1)$ *of entire functions* $f(z)$ *of order* $\rho=\frac{1}{2}$ *and of type* $\leqslant \sigma$ *with*

$$\int_0^{+\infty} |f(x)|^2 x^{\omega}\, dx < +\infty \tag{0.7}$$

coincides with the class of functions

$$f(z)=\int_0^{\sigma} E_{1/2}(-\tau^2 z; \mu)\, \tau^{\mu-1} \varphi(\tau)\, d\tau, \tag{0.8}$$

where $\mu=\omega+\frac{3}{2}$ *and where* $\varphi(\tau)\in L_2(0,\sigma)$ *is arbitrary but uniquely determined by* f.

Finally, the third important result is

THEOREM III.[3] *Let* $\mu\in(\frac{1}{2},\frac{5}{2})$ *and* $\varphi(\tau)\in L_2(0,\sigma)$. *Then the integral* (0.8) *determines an entire function* $f(z)\in W^{2,\omega}_{1/2,\sigma}$ *with*

$$\int_0^{+\infty} |f(r)|^2 r^{\omega}\, dr \asymp \int_0^{\sigma} |\varphi(\tau)|^2\, d\tau, \tag{0.9}$$

where $\omega=\mu-\frac{3}{2}$.[4]

0.2. The paper is organized as follows. In Section 1 we recall a number of preliminary results which are mainly contained in the authors' papers [2, 3, 4]. Among these preliminaries the results on the character of the zeros distribution and on the behaviour of the function

$$\mathscr{E}_{\sigma}(z;\nu)=E_{1/2}(-\sigma^2 z; 1+\nu), \qquad 0\leqslant \nu<2, \tag{0.10}$$

on the semi-axis $(0,+\infty)$ are especially important.

In Section 2 we consider entire functions belonging to the class

$$W^{p,\omega}_{1/2,\sigma}: \int_0^{+\infty} |f(x)|^p x^{\omega}\, dx < +\infty \qquad (1<p<+\infty;\ -1<\omega<p-1\rangle \tag{0.11}$$

[2] See [1, Theorem 6.10].

[3] See [1, Theorem 4.1] for $\rho=\frac{1}{2}$, $-1<\omega<1$ and $\mu=\omega+\frac{3}{2}$.

[4] The sign $\asymp$ denotes that the ratio of the integrals lies between constants independent of x.

and prove our basic Theorems 2.1–2.4 which assert that these functions may be represented by interpolation series with nodes in the zeros $\{\lambda_k\}_1^\infty \subset (0, +\infty)$ of $\mathscr{E}_\sigma(z; \nu)$; there are two kinds of such interpolation series with respect to the two assumptions on the parameter $\nu \in [0, 2)$.

Further, essentially on the basis of Theorems I, II, III above, in Section 3 we pass from the interpolation theorems for classes $W^{2,\omega}_{1/2,\sigma}$ of Section 2 to the construction of the functional systems, which are (after a suitable normalization) the Riesz bases for $L_2(0, \sigma)$. These systems are the natural analogs of the trigonometric systems (0.2) and may be reduced to them for $\omega = \pm\frac{1}{2}$. At the same time, following [2], we are able to write expressions of that and biorthogonal to them systems in terms of $E_{1/2}$ (Theorems 3.1–3.5).

Finally, we note the deep fact that the systems of Section 3 are closely connected with singular boundary value problems for differential operators of fractional order. These boundary value problems have essentially new nature. The authors intend to devote another paper to this question.

1. Preliminaries

1.1. First we introduce some classes of functions and discuss some of their properties.

(a) Let $H^p_\pm$ $(0 < p < +\infty)$ be the class of functions, which are analytic in the half-plane

$$G^\pm \equiv \{z\colon \pm \operatorname{Im} z > 0\} \tag{1.1}$$

and have a finite norm

$$\|\varphi^\pm\|_p = \sup_{0<y<+\infty} \left\{\int_{-\infty}^{+\infty} |\varphi^\pm(x+iy)|^p\, dx\right\}^{1/p} < +\infty. \tag{1.2}$$

Then the following assertions hold (see, for example, [2 or 5]).

1. If $\varphi^\pm(z) \in H^p_\pm$, then the boundary function

$$\lim_{g\to+0} \varphi^\pm(x \pm iy) = \varphi^\pm(x) \in L_p(-\infty, +\infty) \tag{1.3}$$

exists almost everywhere on $(-\infty, +\infty)$. Further

$$\begin{aligned} \|\varphi^\pm\|_p &= \lim_{y\to+0} \|\varphi^\pm(x\pm iy)\|_p, \\ \lim_{y\to+0} &\|\varphi^\pm(x) - \varphi^\pm(x\pm iy)\|_p = 0, \end{aligned} \tag{1.4}$$

and we have the inequality

$$\|\varphi^{\pm}(x \pm iy)\|_p \leqslant \|\varphi^{\pm}\|_p. \tag{1.5}$$

Note that the classes $H^p_{\pm}$ with the introduced norm are Banach spaces provided by $1 \leqslant p < +\infty$.

Let $\varphi(t)$ be a measurable function on $(-\infty, +\infty)$ with finite norm

$$\|\varphi\|_{p,\kappa} = \left(\int_{-\infty}^{+\infty} |\varphi(t)|^p |t+i|^{\kappa}\, dt\right)^{1/p} < +\infty$$
$$(1 < p < +\infty;\ -1 < \kappa < p-1). \tag{1.6}$$

Then the Cauchy type integral

$$\Phi^{\pm}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\varphi(t)}{t-z}\, dt, \qquad z \in G^{\pm}, \tag{1.7}$$

has the following properties (see [2, Sect. 3, Theorems Б, B, Г]):

2. For the functions

$$(z \pm i)^{\kappa/p}, \qquad z \in G^{\pm},$$

in the z-plane with slit $(-i\infty, -i)$ (resp$(i, +i\infty)$), we have

$$(z \pm i)^{\kappa/p}\, \Phi^{\pm}(z) \in H^p_{\pm}, \tag{1.8}$$

whereas

$$\|(x \pm i)^{\kappa/p}\, \Phi^{\pm}(x)\|_p \leqslant C_1 \|\varphi\|_{p,\kappa}. \tag{1.9}$$

(b) Let $W^{p,\kappa}_{\sigma}$ $(1 < p < +\infty;\ -1 < \kappa < +\infty)$ be the set of entire functions $f(z)$ of exponential type $\leqslant \sigma$ with a finite norm

$$\|f\|_{p,\kappa} = \left(\int_{-\infty}^{+\infty} |f(x)|^p |x|^{\kappa}\, dx\right)^{1/p} < +\infty. \tag{1.10}$$

Then the following assertions hold

3. For every function $f(z) \in W^{p,\kappa}_{\sigma}$ $(1 < p < +\infty;\ -1 < \kappa < +\infty)$ we have

$$|f(z)| \leqslant C_2(1+|z|)^{-\kappa/p}\, l^{\sigma|\operatorname{Im} z|}\|f\|_{p,\kappa}, \qquad z \in \mathbb{C}, \tag{1.11}$$

and for all $\gamma \in [0, +\infty)$

$$\|f\|^p_{p,\kappa} \leqslant C_3 \int_{-\infty}^{+\infty} |f(x \pm i\gamma)|^p |x|^{\kappa}\, dx$$
$$\leqslant C_4 \int_{-\infty}^{+\infty} |f(x \pm i\gamma)|^p |x \pm i\gamma|^{\kappa}\, dx. \tag{1.12}$$

Further, it follows from (1.11) that the class $W_\sigma^{p,\kappa}$ with the norm $\|f\|_{p,\kappa}$ is a Banach space (see [2, Lemmas 2.2 and 2.3]).

(c) Finally, we also define the class $W_{1/2,\sigma}^{p,\omega}$ $(1<p<+\infty;$ $-1<\omega<+\infty)$ of entire functions $\Phi(w)$ of order $\frac{1}{2}$ and of type $\leqslant\sigma$ such that the norm (1.13) below is finite

$$\|\Phi\|_{p,\omega}^{*}=\left(\int_0^{+\infty}|\Phi(u)|^p\,u^\omega\,du\right)^{1/p}<+\infty. \tag{1.13}$$

From the above definition we have

4. If $\Phi(w)\in W_{1/2,\sigma}^{p,\omega}$, then

$$f(z)=\Phi(z^2)\in W_\sigma^{p,1+2\omega} \tag{1.14}$$

and

$$\|\Phi\|_{p,\omega}^{*}=\|f\|_{p,1+2\omega}. \tag{1.15}$$

1.2. The entire function of Mittag-Leffler type of order $\rho=\frac{1}{2}$ and of the type 1 is defined via the expansion

$$E_{1/2}(z;\mu)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+2k)}\qquad(\mu>0). \tag{1.16}$$

It is easy to see that for $\mu=1$ or $\mu=2$ we have

$$E_{1/2}(-z^2;1)=\cos z,\qquad E_{1/2}(-z^2;2)=\frac{\sin z}{z}. \tag{1.17}$$

For every $\mu\in(0,3)$ the function $E_{1/2}(z;\mu)$ has the following asymptotical property (see [1, Lemma 3.5]).

5. For $0\leqslant\arg z\leqslant\pi$ respectively for $-\pi\leqslant\arg z\leqslant 0$ and for $-1<\nu<2$ we have

$$E_{1/2}(z;1+\nu)=\frac{1}{2}z^{-\nu/2}\{e^{z^{1/2}}+e^{\pm i\pi\nu}e^{-z^{1/2}}\}+O\left(\frac{1}{z}\right),\qquad |z|\to\infty. \tag{1.18}$$

In particular, for $0<x<+\infty$,

$$E_{1/2}(-x;1+\nu)=x^{-\nu/2}\cos\left(\sqrt{x}-\frac{\pi}{2}\nu\right)+O\left(\frac{1}{x}\right),\qquad x\to+\infty. \tag{1.19}$$

For the later purposes it is convenient to introduce the entire function

$$\mathscr{E}_\sigma(z;\nu)=E_{1/2}(-\sigma^2 z;1+\nu),\qquad \sigma>0, \tag{1.20}$$

of order $\frac{1}{2}$ and of type σ.

Together with the asymptotic formulae from 5, the following result (see [2, Lemmas 1.3′ and 1.3]) is also necessary for us.

6. The two-sided estimation

$$|\mathscr{E}_\sigma((x \pm i)^2; \nu)| \asymp |x \pm i|^{-\nu}, \qquad -\infty < x < +\infty, \tag{1.21}$$

holds. In the angular sectors

$$\mathscr{D}_\pm = \left\{ z: \left| \arg z \mp \frac{\pi}{2} \right| < \frac{\pi}{4}, \ 1 < |z| < +\infty \right\}$$

we have also

$$|\mathscr{E}_\sigma(z^2; \nu)| \asymp (1 + |z|)^{-\nu} e^{\sigma |\mathrm{Im}\, z|}, \qquad z \in \mathscr{D}_+ \cup \mathscr{D}_-. \tag{1.21′}$$

Now we give the necessary information on the distribution of zeros of $\mathscr{E}_\sigma(z; \nu)$. The required properties of these zeros follow from [2, Lemma 1.1].

7. All the zeros of $\mathscr{E}_\sigma(z; \nu)$, $0 \leqslant \nu < 2$, are positive and simple. Let $\{\lambda_k\}_1^\infty (0 < \lambda_k < \lambda_{k+1})$ be the sequence of zeros of $\mathscr{E}_\sigma(z; \nu)$. Then the set $\{x_{\pm k} = \pm\sqrt{\lambda_k}\}_1^\infty$ of all zeros of $\mathscr{E}_\sigma(z^2; \nu)$ lies on the line $\mathrm{Im}\, z = 0$ and is symmetric with respect to the point $z = 0$.

8. The distribution of the zeros of $\mathscr{E}_\sigma(z; \nu)$ is: For $\nu = 0$ (resp. $0 < \nu < 1$) we have

$$\lambda_k = \left(\frac{\pi}{\sigma} k - \frac{\pi}{2\sigma}\right)^2 \qquad \text{resp. } \lambda_k \in \left(\left(\frac{\pi}{\sigma} k - \frac{\pi}{2\sigma}\right)^2, \left(\frac{\pi}{\sigma} k + \frac{\pi}{2\sigma}\right)^2 \right)$$
$$(1 \leqslant k < +\infty). \tag{1.22}$$

For $\nu = 1$ (resp. $1 < \nu < 2$) we have

$$\lambda_k = \left(\frac{\pi}{\sigma} k\right)^2 \qquad \text{resp. } \lambda_k \in \left(\left(\frac{\pi}{\sigma} k\right)^2, \left(\frac{\pi}{\sigma} k + \frac{\pi}{\sigma}\right)^2 \right)$$
$$(1 \leqslant k < +\infty). \tag{1.22′}$$

In particular,

$$\lambda_k \asymp (1 + k)^2, \ |x_k| = |x_{-k}| \asymp 1 + k \qquad (1 \leqslant k < +\infty). \tag{1.23}$$

Moreover, the asymptotical behavior of $\{\lambda_k\}_1^\infty$ is also known.

9. For $0 < \nu < 1$ or $1 < \nu < 2$,

$$\sqrt{\lambda_k} = \frac{\pi}{\sigma} k + \frac{\pi}{2\sigma} (\nu - 1) + O(k^{\nu - 2}) \tag{1.24}$$

and for $\nu=0$ (resp. $\nu=1$),

$$\sqrt{\lambda_k}=\tfrac{\pi}{\sigma}k-\frac{\pi}{2\sigma} \qquad \left(\text{resp. } \sqrt{\lambda_k}=\frac{\pi}{\sigma}k\right)(1\leqslant k<+\infty). \tag{1.25}$$

Using these facts, we prove

LEMMA 1.1. *If* $\{\lambda_k\}_1^\infty$ $(0<\lambda_k<\lambda_{k+1})$ *is the sequence of zeros of* $\mathscr{E}_\sigma(z;\nu)$ $(0\leqslant\nu<2)$, *then for all* $k=1,2,\ldots$ *we have*

$$\mathscr{E}'_\sigma(\lambda_k;\nu)=(2\lambda_k)^{-1}\,\mathscr{E}_\sigma(\lambda_k;\nu-1) \tag{1.26}$$

and

$$|\mathscr{E}_\sigma(\lambda_k;\nu-1)|\asymp(1+k)^{1-\nu}. \tag{1.27}$$

Proof. It is easy to verify that

$$E_{1/2}(-\sigma^2 z;\nu)=\nu E_{1/2}(-\sigma^2 z;1+\nu)\\ -2\sigma^2 zE'_{1/2}(-\sigma^2 z;1+\nu).$$

Now setting $z=\lambda_k$ and using (1.20), we obtain (1.26).

In order to prove (1.27) we use (1.19), where we set $x=\sigma^2\lambda_k$ and replace ν by $\nu-1$. Then we obtain the formula

$$E_{1/2}(-\sigma^2\lambda_k;\nu)=(\sigma^2\lambda_k)^{1/2(1-\nu)}\cos\left(\sigma\sqrt{\lambda_k}+\frac{\pi}{2}(1-\nu)\right)+O\left(\frac{1}{\lambda_k}\right)$$

$$=-(\sigma^2\lambda_k)^{1/2(1-\nu)}\sin\left(\sigma\sqrt{\lambda_k}-\frac{\pi}{2}\nu\right)+O\left(\frac{1}{\lambda_k}\right)$$

$$(k=1,2,\ldots). \tag{1.27'}$$

Now, using the asymptotical formulae (1.24)–(1.25) for $\sqrt{\lambda_k}$ and the double inequality (1.23), we obtain

$$E_{1/2}(-\sigma^2\lambda_k;\nu)=(\sigma^2\lambda_k)^{(1/2)(1-\nu)}\cos[\pi k+O(k^{\nu-2})]+O\left(\frac{1}{\lambda_k}\right). \tag{1.28}$$

But since $0\leqslant\nu<2$ implies $\frac{1}{2}(1-\nu)>-1$, and we have, by the definition (1.20),

$$\mathscr{E}_\sigma(\lambda_k;\nu-1)=E_{1/2}(-\sigma^2\lambda_k;\nu),$$

(1.28) implies the limit relation

$$\lim_{k\to+\infty}(1+k)^{\nu-1}|\mathscr{E}_\sigma(\lambda_k;\nu-1)|=\pi^{1-\nu}$$

and, in particular, (1.27) follows.

LEMMA 1.2. *Let* $1 < p < +\infty$ *and* $-1 < \omega < p-1$. *Then*

1. *For* $\nu \geqslant 0$,

$$\frac{E_{1/2}(-\sigma^2 z; 1+\nu)}{z-\lambda_k} \equiv \frac{\mathscr{E}_\sigma(z; \nu)}{z-\lambda_k} \in W^{p,\omega}_{1/2,\sigma} \qquad (k = 1, 2, \ldots) \tag{1.29}$$

and for $\nu < 2(1+\omega)/p$,

$$\mathscr{E}_\sigma(z; \nu) \equiv E_{1/2}(-\sigma^2 z; 1+\nu) \notin W^{p,\omega}_{1/2,\sigma}. \tag{1.30}$$

2. *For* $\nu > 2(1+\omega)p$,

$$\frac{z\mathscr{E}_\sigma(z; \nu)}{z-\lambda_k} \in W^{p,\omega}_{1/2,\sigma} \qquad (k = 1, 2, \ldots), \tag{1.29'}$$

and for $\nu < 2$,

$$z\mathscr{E}_\sigma(z; \nu) \notin W^{p,\omega}_{1/2,\sigma}. \tag{1.30'}$$

Proof. It is obvious that the function in (1.29) is an entire function of order $\frac{1}{2}$ and of the type σ. Using the asymptotic formula (1.19), we obtain

$$|E_{1/2}(-\sigma^2 x; 1+\nu)| \leqslant C(1+x)^{-\nu/2}, \qquad 0 \leqslant x < +\infty, \tag{1.31}$$

where $C > 0$ is independent of x.

First, it is obvious that

$$\int_0^1 \left| \frac{\mathscr{E}_\sigma(x; \nu)}{x-\lambda_k} \right|^p x^\omega \, dx < +\infty. \tag{1.32}$$

Second, let us see that by (1.31)

$$\left| \frac{\mathscr{E}_\sigma(x; \nu)}{x-\lambda_k} \right|^p x^\omega \leqslant C(k)\, x^{\omega-(1+\nu/2)p}, \qquad 1 \leqslant x < +\infty, \tag{1.32'}$$

where $C(k) > 0$ are constants.

Since $\omega - p(1+\nu/2) < -1$ is provided by $\nu \geqslant 0$ and $-1 < \omega < p-1$, we also have

$$\int_1^{+\infty} \left| \frac{\mathscr{E}_\sigma(x; \nu)}{x-\lambda_k} \right|^p x^\omega \, dx < +\infty. \tag{1.32''}$$

Therefore, by (1.32′) and (1.32″) we can easily deduce the inclusions (1.29) of the lemma. In order to prove (1.30) we observe that (1.19) implies

$$(\sigma^2 x)^{-\nu/2} \left|\cos\left(\sigma\sqrt{x} - \frac{\pi}{2}\nu\right)\right|$$
$$\leqslant |E_{1/2}(-\sigma^2 x;\, 1+\nu)| + \frac{C(\nu, \sigma)}{x},$$
$$1 \leqslant x < +\infty,$$

where $C(\nu, \sigma) > 0$ is independent of x.

Using the well-known inequality $(a+b)^p \leqslant 2^p(a^p + b^p)$, $(0 \leqslant a, b < +\infty)$ we obtain for all $R > 1$,

$$(2\sigma^\nu)^{-p} \int_1^R \left|\cos\left(\sigma\sqrt{x} - \frac{\pi}{2}\nu\right)\right|^p x^{\omega - p\nu/2}\, dx$$
$$\leqslant \int_1^R |E_{1/2}(-\sigma^2 x;\, 1+\nu)|^p x^\omega\, dx + C_p(\nu, \sigma) \int_1^R \frac{dx}{x^{p-\omega}}.$$

In view of $\nu < 2(1+\omega)/p$ and $\omega < p-1$ it is easy to see that in this inequality the left integral becomes to $+\infty$ as $R \to +\infty$, and the second right integral converges. These imply (1.30). The proof of 2 is similar.

1.3. Finally, we also give three important embedding theorems (see [2, Lemmas 3.2 and 3.1]).

10. If $\varphi^\pm(z) \in H^p_\pm$ $(1 < p < +\infty)$, then we have

$$\sum_{R=-\infty}^{+\infty} |\varphi^\pm(x_k \pm i\gamma)|^p \leqslant C_1 \|\varphi^\pm\|_p^p, \tag{1.33}$$

for all $\gamma \in (0, +\infty)$ and $x_0 = 0$.

11. If $f(z) \in W_\sigma^{p,\kappa}$ $(1 < p < +\infty;\ -1 < \kappa < +\infty)$, then

$$\sum_{k=-\infty}^{+\infty} |f(x_k \pm i\gamma)|^p (1+|k|)^\kappa \leqslant C_2 \|f\|_{p,\kappa}^p, \tag{1.34}$$

where $C_j > 0$ $(j = 1, 2, \ldots)$ depend neither on $\varphi^\pm(z)$, nor on $f(z)$.

At last, this and definition 4 of $W^{p,\omega}_{1/2,\sigma}$ imply directly:

12. If $\Phi(w) \in W^{p,\omega}_{1/2,\sigma}$ $(1 < p < +\infty;\ -1 < \omega < +\infty)$, then

$$\sum_{k=1}^{\infty} |\Phi(\lambda_k)|^p (1+k)^{1+2\omega} \leqslant C_3 \{\|\Phi\|^*_{p,\omega}\}^p. \tag{1.35}$$

2. Interpolation Theorems and Bases for $W^{p,\omega}_{1/2,\sigma}$

2.1. Let, as above, $\{\lambda_k = x^2_{\pm k}\}_1^\infty$ be the sequence of the zeros of

$$\mathscr{E}_\sigma(z; \nu) \equiv E_{1/2}(-\sigma^2 z; 1+\nu). \tag{2.1}$$

Recall (see Sects. 1, 7, and 8) that for $0 \leqslant \nu < 2$ all these zeros belong to $(0, +\infty)$. They are simple and

$$\lambda_k \asymp (1+k)^2, \qquad 1 \leqslant k < +\infty. \tag{2.2}$$

(a) Let us consider two formal series

$$\Phi(z) = \sum_{k=1}^{\infty} c_k \frac{2\lambda_k \mathscr{E}_\sigma(z; \nu)}{\mathscr{E}_\sigma(\lambda_k; \nu-1)(z-\lambda_k)}, \tag{2.3}$$

$$\Psi(z) = d_0 \Gamma(1+\nu)\, \mathscr{E}_\sigma(z; \nu) + \sum_{k=1}^{\infty} d_k \frac{2z\mathscr{E}_\sigma(z; \nu)}{\mathscr{E}_\sigma(\lambda_k; \nu-1)(z-\lambda_k)}, \tag{2.4}$$

where $\{c_k\}_1^\infty$ and $\{d_k\}_0^\infty$ are complex sequences.

In view of (1.26) of Lemma 1.1 the sums of these series satisfy the interpolation conditions

$$\Phi(\lambda_k) = c_k (1 \leqslant k < +\infty), \qquad \Psi(\lambda_k) = d_k (\lambda_0 = 0, 0 \leqslant k < +\infty). \tag{2.5}$$

First, we are interested in conditions under which $\Phi(z)$ and $\Psi(z)$ belong to $W^{p,\omega}_{1/2,\sigma}$.

Let $l^{p,\kappa}$ $(1 < p < +\infty;\ -1 < \kappa < +\infty)$ be a Banach space of sequences $\{a_k\}_1^\infty \subset \mathbb{C}$ with finite norm

$$\|\{a_k\}\|_{p,\kappa} = \left(\sum_{k=1}^{\infty} |a_k|^p (1+k)^\kappa \right)^{1/p} < +\infty. \tag{2.6}$$

Lemma 2.1. *Let* $\{c_k\}_1^\infty \in l^{p,1+2\omega}$, $\{d_k\}_0^\infty \in l^{p,1+2\omega}$ $(1 < p < +\infty;\ -1 < \omega < p-1)$. *Then for*

$$0 \leqslant \nu < \frac{2(1+\omega)}{p} \qquad (\text{resp. } 0 \leqslant \nu < 2), \tag{2.7}$$

the series (2.3) (resp. (2.4)) *converges absolutely and uniformly on every compact subset of complex plane. The sum* $\Phi(z)$ (resp. $\Psi(z)$) *of this series is an entire function, which satisfies the interpolation condition* (2.5).

Proof. Let $K \subset \mathbb{C}$ be a compact subset without any zero $\{\lambda_k\}_1^\infty$ of $\mathscr{E}_\sigma(z; \nu)$.

For all $1 \leqslant n < m < +\infty$ and all segments of the series (2.3) we have

$$|\Phi_{n,m}(z)| = \left| \sum_{k=n}^{m} c_k \frac{2\lambda_k \mathscr{E}_\sigma(z; \nu)}{\mathscr{E}_\sigma(\lambda_k; \nu - 1)(z - \lambda_k)} \right|$$
$$\leqslant B_1 \sum_{k=n}^{m} \frac{|c_k|}{|\mathscr{E}_\sigma(\lambda_k; \nu - 1)|}, \qquad z \in K,$$

where

$$B_1 = \sup_{z \in K} \max_{1 \leqslant k < +\infty} \left| \frac{2\lambda_k \mathscr{E}_\sigma(z; \nu)}{z - \lambda_k} \right|$$

is, of course, finite. Formula (1.27) of Lemma 1.1 and the Hölder inequality imply

$$|\Phi_{n,m}(z)| \leqslant B_2 \sum_{k=n}^{m} |c_k|(1+k)^{\nu - 1}$$
$$\leqslant B_2 \| \{c_k\}_n^m \|_{p,1+2\omega} \left\{ \sum_{k=n}^{m} (1+k)^{q(\nu - 1 - (1+2\omega)/p)} \right\}^{1/q}, \quad z \in K,$$

where $q = p/(p-1)$.

But for $\gamma < 2(1+\omega)/p$ we have $q(\gamma - 1 - (1+2\omega)/p) < \gamma$ and hence the second sum above is a segment of a convergent series. Therefore for all $1 \leqslant n < m < +\infty$ we obtain the estimation

$$|\Phi_{n,m}(z)| \leqslant B_3 \| \{c_k\}_n^m \|_{p,1+2\omega}, \qquad z \in K,$$

where $B_3 > 0$ is independent of n, m.

Now in view of the condition $\{c_k\}_1^\infty \in l^{p,1+2\omega}$ via passage to the limit we obtain the required assertions about the convergence of the series (2.3) and properties of its sum $\Phi(z)$.

The assertion about the convergent nature of (2.4) provided by $0 \leqslant \nu < 2$ may be proved similarly.

(b) Now we prove the basic interpolation theorems.

THEOREM 2.1. *Let* $\{c_k\}_1^\infty \in l^{p,1+2\omega}$ $(1 < p < +\infty;\ -1 < \omega < p-1)$ *and let the parameter* ν $(0 \leqslant \nu < 2)$ *satisfy the condition*

$$\frac{2(1+\omega)}{p} - 1 < \nu < \frac{2(1+\omega)}{p}. \tag{2.8}$$

Then the series

$$\Phi(z)=\sum_{k=1}^{\infty} c_k \frac{2\lambda_k \mathscr{E}_\sigma(z;\nu)}{\mathscr{E}_\sigma(\lambda_k;\nu-1)(z-\lambda_k)} \tag{2.9}$$

converges to an entire function $\Phi(z)$ *in* $W^{p,\omega}_{1/2,\sigma}$ *metric, whereas*

$$\Phi(\lambda_k)=c_k(1\leqslant k<+\infty), \qquad \|\Phi\|^*_{p,\omega}\asymp\|\{c_k\}_1^\infty\|_{p,1+2\omega}. \tag{2.10}$$

Proof. As above, put

$$\Phi_{n,m}(z)=\sum_{k=n}^{m} c_k \frac{2\lambda_k \mathscr{E}_\sigma(z;\nu)}{\mathscr{E}_\sigma(z;\nu-1)(z-\lambda_k)}.$$

Then Lemma 1.2(1) implies that $\Phi_{n,m}(z)\in W^{p,\omega}_{1/2,\sigma}$, whereas

$$\{\|\Phi_{n,m}\|^*_{p,\omega}\}^p=\int_0^{+\infty}|\Phi_{n,m}(x)|^p x^\omega\,dx$$

$$=\int_{-\infty}^{+\infty}|\Phi_{n,m}(x^2)|^p|x|^{1+2\omega}\,dx.$$

Using the inequality (1.12) of 3 (Sect. 1) and putting

$$\varphi_{n,m}(z)=\sum_{k=n}^{m}\frac{2c_k\lambda_k}{\mathscr{E}_\sigma(\lambda_k;\nu-1)(z^2-\lambda_k)} \tag{2.11}$$

we obtain from the inequalities (1.21) of Section 1(6)

$$\begin{aligned}\{\|\Phi_{n,m}\|^*_{p,\omega}\}^p&=\|\Phi_{n,m}(x^2)\|^p_{p,1+2\omega}\\ &\leqslant B_1\int_{-\infty}^{+\infty}|\Phi_{n,m}[(x+i)^2]|^p|x+i|^{1+2\omega}\,dx\\ &=B_1\int_{-\infty}^{+\infty}|\mathscr{E}_\sigma[(x+i)^2,\nu]\,\varphi_{n,m}(x+i)|^p|x+i|^{1+2\omega}\,dx\\ &\leqslant B_2\int_{-\infty}^{+\infty}|\varphi_{n,m}(x+i)|^p|x+i|^{\omega_1}\,dx,\end{aligned} \tag{2.12}$$

where $\omega_1=1+2\omega-p\nu$ belongs to $(-1,p-1)$ in view of (2.8).

But the Hahn–Banach theorem implies

$$\|\varphi_{n,m}(x+i)\|_{p,\omega_1}=\sup_{\|g\|_q\leqslant 1}\left\{\left|\int_{-\infty}^{+\infty}\varphi_{n,m}(x+i)\,g(x)|x+i|^{\omega_1/p}\,dx\right|\right\},$$

and hence there exists a function $G_0(x)$ such that $G_0(x)|x+i|^{-(\omega_1/p)} \in L_q(-\infty, +\infty)$ with norm

$$\|G_0\|_{q,\omega_2} = \left(\int_{-\infty}^{+\infty} |G_0(x)|^q |x+i|^{\omega_2}\, dx\right)^{1/q} \leqslant 1,$$

where $\omega_2 = -q\omega_1/p$ and $-1 < \omega_2 < q-1$ again, whereas

$$\|\varphi_{n,m}(x+i)\|_{p,\omega_1} \leqslant 2\left|\int_{-\infty}^{+\infty} \varphi_{n,m}(x+i)\, G_0(x)\, dx\right|. \tag{2.13}$$

Now let us consider the Cauchy type integral

$$G(z) = \int_{-\infty}^{+\infty} \frac{G_0(t)}{t-z}\, dt, \qquad \operatorname{Im} z < 0,$$

which has the properties

$$G(z)(z-i)^{\omega_2/q} \in H^q\ , \qquad \|G(x)(x-i)^{\omega_2/q}\|_q \leqslant B_3, \tag{2.14}$$

by 2 (Sect. 1).

If we calculate $\varphi_{n,m}(x+i)$ by (2.11), substitute this value in (2.13), and recall that $\lambda_k = x_k^2$, then we obtain via calculations of residues

$$\begin{aligned}
\|\varphi_{n,m}(x+i)\|_{p,\omega_1} &\leqslant 2\left|\sum_{k=n}^{m} \frac{2c_k\lambda_k}{\mathscr{E}_\sigma(\lambda_k;\nu-1)} \int_{-\infty}^{+\infty} \frac{G_0(x)\,dx}{(x+i)^2 - x_k^2}\right| \\
&= \left|\sum_{k=n}^{m} \frac{4c_k\sqrt{\lambda_k}}{\mathscr{E}_\sigma(\lambda_k;\nu-1)}\,[G(x_k-i) - G(-x_k-i)]\right| \\
&\leqslant \left|\sum_{k=n}^{m} \frac{4c_k\sqrt{\lambda_k}\,G(x_k-i)}{\mathscr{E}_\sigma(\lambda_k;\nu-1)}\right| + \left|\sum_{k=n}^{m} \frac{4c_k\sqrt{\lambda_k}\,G(-x_k-i)}{\mathscr{E}_\sigma(\lambda_k;\nu-1)}\right| \\
&\equiv I_{n,m}^{(1)} + I_{n,m}^{(2)}.
\end{aligned} \tag{2.15}$$

In order to estimate $I_{n,m}^{(1)}$ we use the Hölder inequality and note that

$$x_k = \sqrt{\lambda_k} \asymp 1+k, \quad |\mathscr{E}_\sigma(\lambda_k;\nu-1)| \asymp (1+k)^{1-\nu}; \quad k \geqslant 1.$$

Then by (2.15) we obtain

$$I_{n,m}^{(1)} \leqslant B_4\|\{c_k\}_n^m\|_{p,1+2\omega} \left\{\sum_{k=n}^{m} \left|G_0(x_k-i)\right|^q (1+k)^{\omega_2}\right\}^{1/q},$$

and therefore by the embedding Theorem 10 (Sect. 1) and by the second inequality of (2.14) we obtain the estimation

$$I_{n,m}^{(1)} \leqslant B_5\|\{c_k\}_n^m\|_{p,1+2\omega}, \qquad 1 \leqslant n < m < +\infty.$$

By a completely analogous consideration we can also obtain the estimation

$$I^{(2)}_{n,m} \leqslant B_6 \|\{c_k\}_n^m\|_{p,1+2\omega}, \qquad 1 \leqslant n < m < +\infty,$$

of the second summand of (2.15).

In view of (2.13), (2.15), and (2.12) these estimations imply

$$\|\Phi_{n,m}\|^*_{p,\omega} \leqslant B_7 \|\{c_k\}_n^m\|_{p,1+2\omega}, \qquad 1 \leqslant n < m < +\infty, \tag{2.16}$$

where $B_j > 0$ $(j = 1, 2,...)$ are constants independent of n, m.

Since $\{c_k\} \in l^{p,1+2\omega}$, by (2.16) we have $\|\Phi_{n,m}\|^*_{p,\omega} \to 0$ as $n, m \to +\infty$. Therefore the series (2.9) converges to $\Phi(z) \in W^{p,\omega}_{1/2,\sigma}$ in $W^{p,\omega}_{1/2,\sigma}$ norm, and

$$\|\Phi\|^*_{p,\omega} \leqslant B_8 \|\{c_k\}\|_{p,1+2\omega}. \tag{2.17}$$

Since the interpolation properties of $\Phi(z)$ were established above and the opposite inequality to (2.17) follows from 12 (Sect. 1), the proof is completed.

THEOREM 2.2. *Let* $\{d_k\}_0^\infty \in l^{p,1+2\omega}$ $(1 < p < +\infty;\ -1 < \omega < p-1)$ *and the parameter* ν $(0 \leqslant \nu < 2)$ *satisfy the condition*

$$\frac{2(1+\omega)}{p} < \nu < 1 + \frac{2(1+\omega)}{p}. \tag{2.18}$$

Then the series

$$\Psi(z) = d_0 \Gamma(1+\nu)\, \mathscr{E}_\sigma(z;\nu) + \sum_{k=1}^{\infty} d_k \frac{2z\mathscr{E}_\sigma(z;\nu)}{\mathscr{E}_\sigma(\lambda_k;\nu-1)(z-\lambda_k)} \tag{2.19}$$

converges to an entire function $\Psi(z)$ *with respect to* $W^{p,\omega}_{1/2,\sigma}$ *metric, and*

$$\Psi(\lambda_k) = d_k\ (0 \leqslant k < +\infty), \qquad \|\Psi\|^*_{p,\omega} \asymp \|\{d_k\}_0^\infty\|_{p,1+2\omega}. \tag{2.20}$$

Proof. This theorem is similar to Theorem 2.1, and therefore we shall give only a sketch of the proof.

It follows from Lemma 1.2(2) that for all $1 \leqslant n < m < +\infty$

$$\Psi_{n,m}(z) = \sum_{k=n}^{m} d_k \frac{2z\mathscr{E}_\sigma(z;\nu)}{\mathscr{E}_\sigma(\lambda_k;\nu-1)(z-\lambda_k)} \in W^{p,\omega}_{1/2,\sigma},$$

and hence $\Psi_{n,m}(z^2) \in W^{p,1+2\omega}_\sigma$. Further, putting

$$\Psi_{n,m}(z) = \sum_{k=n}^{m} \frac{2z\, d_k}{\mathscr{E}_\sigma(\lambda_k;\nu-1)(z^2-\lambda_k)},$$

and making use of inequality (1.21), we have

$$\{\|\Psi_{n,m}\|^{*}_{p,\omega}\}^{p} \leqslant \mathscr{D}_1 \int_{-\infty}^{+\infty} |\Psi_{n,m}[(x+i)^2]|^{p} |x+i|^{1+2\omega}\, dx. \qquad (2.21)$$

Now, by repeating the arguments and calculations used in the proof of Theorem 2.1, we may conclude that there exists a function $H_0(z)$ with $H_0(x)|x+i|^{-\omega_3/p} \in L_q(-\infty, +\infty)$, where $\omega_3 = 2\omega + 1 + p(1-\nu)$ and by (2.18) we have $-1 < \omega_3 < p-1$.

Furthermore

$$\|\Psi_{n,m}(x+i)\|_{p,\omega_3} \leqslant 2 \left| \int_{-\infty}^{+\infty} \Psi_{n,m}(x+i)\, H_0(x)\, dx \right|. \qquad (2.22)$$

Further, if we consider the Cauchy type integral

$$H(z) = \int_{-\infty}^{+\infty} \frac{H_0(t)}{t-z}\, dt, \qquad \operatorname{Im} z < 0,$$

then we obtain the estimations

$$\begin{aligned} \|\Psi_{n,m}(x+i)\|_{p,\omega_3} &\leqslant \sum_{k=n}^{m} \frac{4d_k}{\mathscr{E}_\sigma(\lambda_k;\nu-1)} \int_{-\infty}^{+\infty} \frac{(x+i)\, H_0(x)}{(x+i)^2 - x_k^2}\, dx \Bigg| \\ &\leqslant \sum_{k=n}^{m} \frac{8d_k}{\mathscr{E}_\sigma(\lambda_k;\nu-1)} H(x_k - i) \Bigg| \\ &\quad + \left| \sum_{k=n}^{m} \frac{8d_k}{\mathscr{E}_\sigma(\lambda_k;\nu-1)} H(-x_k - i) \right| \\ &\equiv I^{(1)}_{n,m} + I^{(2)}_{(n,m)}. \end{aligned}$$

Again, using the relation $|\mathscr{E}_\sigma(\lambda_k;\nu-1)| \asymp (1+k)^{1-\nu}$ we may apply the Hölder inequality and the embedding Theorem 11 (Sect. 1) to obtain

$$\begin{aligned} I^{(1)}_{n,m} &\leqslant \mathscr{D}_2 \|\{d_k\}_n^m\|_{p,1+2\omega} \left\{ \sum_{k=n}^{m} |H(x_k - i)|^{q} (1+k)^{\omega_4} \right\}^{1/q} \\ &\leqslant \mathscr{D}_3 \|\{d_k\}_n^m\|_{p,1+2\omega} \qquad (\omega_4 = -(q/p)\,\omega_3). \end{aligned}$$

Similarly we also obtain

$$I^{(2)}_{n,m} \leqslant \mathscr{D}_4 \|\{d_k\}_n^m\|_{p,1+2\omega},$$

and from it (more exactly, from (2.21) and (2.22)), according to the two latest estimates, we obtain

$$\|\Psi_{n,m}\|^{*}_{p,\omega} \leqslant \mathscr{D}_5 \|\{d_k\}_n^m\|_{p,1+2\omega} \qquad (0 \leqslant n < m < +\infty),$$

where all $\mathscr{D}_j > 0$ ($j = 1, 2, \ldots$) above are constants independent of n, m.

Since $\{d_k\}_0^\infty \in l^{p,1+2\omega}$, we have $\|\Psi_{n,m}\|_{p,\omega}^* \to 0$ as $n, m \to +\infty$. Therefore the series (2.19) converges with respect to the $W_{1/2,\sigma}^{p,\omega}$ metric and hence the proof is completed.

2.3. Here we first establish the converses of Theorems 2.1 and 2.2.

THEOREM 2.3. *Assuming*

$$\frac{2(1+\omega)}{p} - 1 < \nu < \frac{2(1+\omega)}{p} \qquad (0 \leqslant \nu < 2),$$

every function $\Phi(z) \in W_{1/2,\sigma}^{p,\omega}$ $(1 < p < +\infty;\ -1 < \omega < p-1)$ *can be expanded in the series*

$$\Phi(z) = \sum_{k=1}^{\infty} \Phi(\lambda_k) \frac{2\lambda_k \mathscr{E}_\sigma(z;\nu)}{\mathscr{E}_\sigma(\lambda_k;\nu-1)(z-\lambda_k)}, \tag{2.23}$$

which converges uniformly on every compact subset of $\mathbb{C}$ *with respect to the* $W_{1/2,\sigma}^{p,\omega}$ *norm.*

Proof. The assertion about the nature of the convergence of the series (2.23) follows from the embedding Theorem 12 (Sect. 1), Lemma 2.1, and Theorem 2.1. It is obvious that the function

$$\Phi_*(z) \equiv \Phi(z) - \sum_{k=1}^{\infty} \Phi(\lambda_k) \frac{2\lambda_k \mathscr{E}_\sigma(z;\nu)}{\mathscr{E}_\sigma(\lambda_k;\nu-1)(z-\lambda_k)}$$

belongs to $W_{1/2,\sigma}^{p,\omega}$ and

$$\Phi_*(\lambda_k) = 0 \qquad (1 \leqslant k < +\infty). \tag{2.24}$$

Now let us define the function

$$\Omega(z) = \frac{\Phi_*(z^2)}{\mathscr{E}_\sigma(z^2;\nu)}, \tag{2.25}$$

which, by (2.24), is an entire function of the exponential type $\leqslant \sigma$ (see, for example, [6, Chap. 1, Sect. 9]).

In view of the estimation from 6 (Sect. 1) we have in the angular sectors $\mathscr{D}_\pm = \{z\colon |\arg z \mp \pi/2| < \pi/4,\ 1 < |z| < +\infty\}$ the estimates

$$|\mathscr{E}_\sigma(z^2;\nu)| \asymp (1+|z|)^{-\nu} e^{\sigma|\operatorname{Im} z|}, \qquad z \in \bar{\mathscr{D}}_\pm. \tag{2.26}$$

On the other hand, since $\Phi_*(z) \in W_{1/2,\sigma}^{p,\omega}$, we have $\Phi_*(z^2) \in W_\sigma^{p,1+2\omega}$ and hence the inequality (1.11) of 3 (Sect. 1) implies

$$|\Phi_*(z^2)| \leqslant C_1(1+|z|)^{-(1+2\omega)/p} e^{\sigma|\operatorname{Im} z|} \|\Phi_*\|_{p,\omega}^*. \tag{2.27}$$

Now (2.25), (2.26), and (2.27) imply the estimation

$$|\Omega(z)| \leqslant C_2(1+|z|)^{\nu-(2\omega+1)/p}, \qquad z \in \bar{\mathscr{D}}_{\pm},$$

and since $\nu < 2(1+\omega)/p$, we have therefore

$$|\Omega(z)| \leqslant C_3(1+|z|)^{1/p}, \qquad z \in \bar{\mathscr{D}}_{\pm}, \tag{2.28}$$

where $C_j > 0$ $(j = 1, 2, 3)$ are constants independent of z.

Since $\Omega(z)$ is of exponential type and $|\Omega(z)|\,|1+|z||)^{-1/p}$ is bounded at the boundaries of the domains $\mathscr{D}_{\pm}$ by (2.28), it follows from the Phragmén–Lindelöf principle that it is bounded in $\mathbb{C}$. Hence we have $\Omega(z) \equiv a_0$, i.e.,

$$\Phi_*(z) \equiv a_0 \mathscr{E}_\sigma(z; \nu). \tag{2.29}$$

However, $\Phi_*(z) \in W^{p,\omega}_{1/2,\sigma}$ and by (1.30) (see Lemma 1.2(1)), $\mathscr{E}_\sigma(z; \nu) \notin W^{p,\omega}_{1/2,\sigma}$. Therefore the identity (2.29) leads to contradiction unless $\Phi_*(z) \equiv a_0 = 0$, and the theorem is proved.

THEOREM 2.4. *Assuming*

$$\frac{2(1+\omega)}{p} < \nu < 1 + \frac{2(1+\omega)}{p} \qquad (0 \leqslant \nu < 2),$$

every function $\Psi(z) \in W^{p,\omega}_{1/2,\sigma}$ $(1 < p < +\infty;\ -1 < \omega < p-1)$ *may be expanded in the series*

$$\Psi(z) = \Psi(0)\,\Gamma(1+\nu)\,\mathscr{E}_\sigma(z; \nu) + \sum_{k=1}^{\infty} \Psi(\lambda_k) \frac{2z\mathscr{E}_\sigma(z; \nu)}{\mathscr{E}_\sigma(z; \nu-1)(z-\lambda_k)}, \tag{2.30}$$

which converges uniformly on every compact subset of $\mathbb{C}$ with respect to the $W^{p,\omega}_{1/2,\sigma}$ norm.

Proof. The convergence nature can be established as in Theorem 2.3. Let us introduce a function

$$\Psi_*(z) \equiv \Psi(z) - \Psi(0)\,\Gamma(1+\nu)\,\mathscr{E}_\sigma(z; \nu) - \sum_{k=1}^{\infty} \Psi(\lambda_k) \frac{2z\mathscr{E}_\sigma(z; \nu)}{\mathscr{E}_\sigma(\lambda_k; \nu-1)(z-\lambda_k)},$$

belonging to $W^{p,\omega}_{1/2,\sigma}$ and satisfying the conditions

$$\Psi_*(\lambda_k) = 0 \qquad (0 \leqslant k < +\infty;\ \lambda_0 = 0).$$

If we consider the entire function

$$\Omega(z)=\frac{\Psi_*(z^2)}{z\mathscr{E}_\sigma(z^2;\nu)}$$

then using the Lemma 1.2(2), the arguments similar to those of Theorem 2.3 imply $\Psi_*(z)\equiv 0$, and the theorem is proved.

The above theorems imply the following uniqueness theorem.

THEOREM 2.5. *Let* $\Phi(z)\in W^{p,\omega}_{1/2,\sigma}$ $(1<p<+\infty;\ -1<\omega<p-1)$ *and let* $\{\lambda_k\}_1^\infty$ *be the zeros of*

$$\mathscr{E}_\sigma(z;\nu)=E_{1/2}(-\sigma^2 z;1+\nu).$$

Then the identity $\Phi(z)\equiv 0$ *follows from*

1. $\Phi(\lambda_k=0$ $(1\leqslant k<+\infty)$ *provided*

$$\frac{2(1+\omega)}{p}-1<\nu<\frac{2(1+\omega)}{p}\qquad(0\leqslant\nu<2)$$

or from

2. $\Phi(\lambda_k)=0$ $(0\leqslant k<+\infty)$, *where* $\lambda_0=0$, *provided*

$$\frac{2(1+\omega)}{p}<\nu<1+\frac{2(1+\omega)}{p}\qquad(0\leqslant\nu<2).$$

On the basis of Theorems 2.1–2.4 *we also obtain the following general result.*

THEOREM 2.6. 1. *Let* $\{\lambda_k\}_1^\infty$ *be the zeros of* $E_{1/2}(-\sigma^2 z;1+\nu)$ $(0\leqslant\nu<2)$ *and*

$$\frac{2(1+\omega)}{p}-1<\nu<\frac{2(1+\omega)}{p}.\tag{2.31}$$

Then the series of the form (2.9) *establish a one to one continuous mapping of the space of all sequences* $\{C_k\}_1^\infty\in l^{p,1+2\omega}$ $(1<p<+\infty;\ -1<\omega<p-1)$ *onto the space of all entire functions* $\Phi(z)\in W^{p,\omega}_{1/2,\sigma}$, *whereas*

$$\Phi(\lambda_k)=c_k(1\leqslant k<+\infty),\qquad \|\Phi\|^*_{p,\omega}\asymp\|\{c_k\}_1^\infty\|_{p,1+2\omega}.$$

2. *Let* $\{\lambda_k\}_1^\infty$ *be as above but in the case*

$$\frac{2(1+\omega)}{p}<\nu<1+\frac{2(1+\omega)}{p}.\tag{2.32}$$

Then the series of the form (2.19) of Theorem 2.2 establish a one to one continuous mapping of the space of all sequences $\{d_k\}_0^\infty \in l^{p,1+2\omega}$ onto the space of all entire functions $\Psi(z) \in W^{p,\omega}_{1/2,\sigma}$, whereas

$$\Psi(\lambda_k) = d_k (0 \leqslant k < +\infty), \qquad \|\Psi\|^*_{p,\omega} \asymp \|\{d_k\}_0^\infty\|_{p,1+2\omega},$$

where $\lambda_0 = 0$.

For later purposes it is expedient to introduce brief notations for the systems of entire functions, which play a role in the above expansion theorems. Namely, put

$$\omega_k(z) = \frac{2\lambda_k E_{1/2}(-\sigma^2 z; 1+\nu)}{E_{1/2}(-\sigma^2\lambda_k; \nu)(z-\lambda_k)} = \frac{2\lambda_k \mathscr{E}_\sigma(z; \nu)}{\mathscr{E}_\sigma(\lambda_k; \nu-1)(z-\lambda_k)} \tag{2.33}$$

$(k = 1, 2, ...)$ and

$$\begin{aligned} \omega_0^*(z) &= \Gamma(1+\nu)\, E_{1/2}(-\sigma^2 z; 1+\nu) = \Gamma(1+\nu)\, \mathscr{E}_\sigma(z; \nu), \\ \omega_k^*(z) &= \frac{z}{\lambda_k}\, \omega_k(z) \qquad (k = 1, 2, ...). \end{aligned} \tag{2.34}$$

Then we have the two functional systems

$$\{\omega_k(z)\}_1^\infty \subset W^{p,\omega}_{1/2,\sigma} \qquad \text{and} \qquad \{\omega_k^*(z)\}_0^\infty \subset W^{p,\omega}_{1/2,\sigma} \tag{2.35}$$

and for $p = 2$, Theorem 2.6 may be reformulated as

THEOREM 2.7. 1. *Let* $\{\lambda_k\}_1^\infty$ *be the zeros of* $E_{1/2}(-\sigma^2 z; 1+\nu)$ *where*

$$\nu \in (\omega, 1+\omega) \cap [0, 2), \qquad -1 < \omega < 1. \tag{2.36}$$

Then, after a suitable renormalization, the system $\{\omega_k(z)\}_1^\infty$ *forms a Riesz basis for* $W^{2,\omega}_{1/2,\sigma}$.

2. *Let* $\{\lambda_k\}_0^\infty$ *be the zeros of*

$$zE_{1/2}(-\sigma^2 z; 1+\nu) \qquad (\text{i.e., } \lambda_0 = 0),$$

where

$$\nu \in (1+\omega, 2+\omega) \cap [0, 2), \qquad -1 < \omega < 1. \tag{2.37}$$

Then, after a suitable renormalization, the system $\{\omega_k^*(z)\}_0^\infty$ *forms a Riesz basis for* $W^{2,\omega}_{1/2,\sigma}$.

In the end of the paragraph we show three particular cases of the choice of ν, when the sequences $\{\lambda_k\}$, the functional systems (2.33)–(2.34), and hence the interpolation series of Theorems 2.1–2.4 and Theorem 2.6 will take the simplest form.

1. If $-1<\omega<p/2-1$ $(1<p<+\infty)$ and $\nu=0$, then it is easy to verify that condition (2.31) of Theorem 2.6(1) is satisfied. In this case the expansion (2.23) (see Theorem 2.3) becomes

$$\Phi(z)=\frac{2\pi}{\sigma^2}\sum_{k=1}^{\infty}(-1)^{k-1}\Phi\left(\left[\frac{\pi}{\sigma}\left(k-\frac{1}{2}\right)\right]^2\right)\frac{(k-1/2)\cos\sigma\sqrt{z}}{z-[(\pi/\sigma)(k-1/2)]^2}. \tag{2.38}$$

2. If $(p/2)-1<\omega<p-1$ and $\nu=1$, then condition (2.31) of Theorem 2.6(1) is satisfied again, and Theorem 2.3 implies the expansion

$$\Phi(z)=2\left(\frac{\pi}{\sigma}\right)^2\sum_{k=1}^{\infty}(-1)^k\frac{k^2(\sigma\sqrt{z})^{-1}\sin\sigma\sqrt{z}}{z-((\pi/\sigma)k)^2}. \tag{2.39}$$

3. If $-1<\omega<(p/2)-1$ and $\nu=1$, then it is easy to verify that condition (2.32) of Theorem 2.6(2) is satisfied. Hence the expansion (2.30) (see Theorem 2.4) can be written in a form

$$\begin{aligned}\Psi(z)=\Psi(0)\frac{\sin\sigma\sqrt{z}}{\sigma\sqrt{z}}&\\ +\frac{2}{\sigma}\sum_{k=1}^{\infty}(-1)^k\Psi\left(\left(\frac{\pi}{\sigma}k\right)^2\right)\frac{\sqrt{z}\sin\sigma\sqrt{z}}{z-(\pi/\sigma)k)^2}.&\end{aligned} \tag{2.40}$$

Of course, the above estimations of the norms are also valid for these particular expansions.

3. Biorthogonal Bases for $L_2(0,\sigma)$

3.1. First we prove three lemmas.

Lemma 3.1.[5] *For all* $0<\alpha<\beta<+\infty$ *and for arbitrary complex* z, λ *there the formula*

$$\begin{aligned}I(z;\lambda)&\equiv\int_0^\sigma E_{1/2}(-z\tau^2;\alpha)\,\tau^{\alpha-1}E_{1/2}(-\lambda(\sigma-\tau)^2;\beta)(\sigma-\tau)^{\beta-1}\,d\tau\\ &=\frac{E_{1/2}(-\sigma^2z;\alpha+\beta-2)-E_{1/2}(-\sigma^2\lambda;\alpha+\beta-2)}{\lambda-z}\sigma^{\alpha+\beta-3}\end{aligned} \tag{3.1}$$

is valid.

[5] See [1, Chap. 3], particular case of the formula (1.21).

Proof. Using the expansions

$$E_{1/2}(-z\tau^2;\alpha)=\sum_{n=0}^{\infty}(-1)^n\frac{z^n\tau^{2n}}{\Gamma(\alpha+2n)},$$

$$E_{1/2}(-\lambda(\sigma-\tau)^2;\beta)=\sum_{m=0}^{\infty}(-1)^m\frac{\lambda^m(\sigma-\tau)^{2m}}{\Gamma(\beta+2m)},$$

we obtain for $\lambda\neq z$,

$$\begin{aligned}I(z;\lambda)&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\frac{z^n\lambda^m}{\Gamma(\alpha+2n)\,\Gamma(\beta+2m)}\int_0^{\sigma}\tau^{2n+\alpha-1}(\sigma-\tau)^{2m+\beta-1)}\,d\tau\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\frac{z^n\lambda^m\sigma^{2(n+m)+\alpha+\beta-1}}{\Gamma(\alpha+\beta+2(n+m))}\\&=\sigma^{\alpha+\beta-1}\sum_{k=0}^{\infty}\sum_{n=0}^{k}(-1)^k\frac{z^n\lambda^{k-n}\sigma^{2k}}{\Gamma(\alpha+\beta+2k)}\\&=\frac{\sigma^{\alpha+\beta-1}}{z-\lambda}\sum_{k=0}^{\infty}(-1)^k\frac{\sigma^{2k}(z^{k+1}-\lambda^{k+1})}{\Gamma(\alpha+\beta+2k)}\\&=\frac{zE_{1/2}(-\sigma^2z;\alpha+\beta)-\lambda E_{1/2}(-\sigma^2\lambda;\alpha+\beta)}{z-\lambda}\,\sigma^{\alpha+\beta-1}.\end{aligned}\tag{3.1'}$$

Hence, using the obvious identity

$$zE_{1/2}(z;\mu+2)=E_{1/2}(z;\mu)-1/\Gamma(\mu),$$

we obtain formula (3.1) for all $\lambda\in\mathbb{C}$.

In Section 2 we defined the two systems $\{\omega_k(z)\}_1^{\infty}$ and $\{\omega_k^*(z)\}_0^{\infty}$ of entire functions where

$$\begin{aligned}\omega_k(z)&=\frac{2\lambda_k E_{1/2}(-\sigma^2z;1+\nu)}{E_{1/2}(-\sigma^2\lambda_k;\nu)(z-\lambda_k)}\\&=\frac{2\lambda_k\mathscr{E}_\sigma(z;\nu)}{\mathscr{E}_\sigma(\lambda_k;\nu-1)(z-\lambda_k)}\end{aligned}\tag{3.2}$$

$(k=1, 2,...)$, whereas

$$\omega_k(\lambda_n)=\delta_{k,n}\qquad (k, n=1, 2,...),\tag{3.2'}$$

and

$$\begin{aligned}\omega_0^*(z)&=\Gamma(1+\nu)\,E_{1/2}(-\sigma^2z;1+\nu)=\Gamma(1+\nu)\,\mathscr{E}_\sigma(z;\nu),\\ \omega_k^*(z)&=\frac{z}{\lambda_k}\,\omega_k(z)\qquad (k=1, 2,...),\end{aligned}\tag{3.3}$$

whereas

$$\omega_k^*(\lambda_n) = \delta_{k,n} \qquad (k, n = 0, 1, 2, \ldots). \tag{3.3'}$$

According to Lemma 1.2 we have for $p = 2$ and $-1 < \omega < 1$:

1. If $\nu \in (\omega, 1 + \omega) \cap [0, 2)$, then the system of functions (3.2) lies in $W^{2,\omega}_{1/2,\sigma}$. Therefore by Theorem II (Sect. 0) these functions admit an integral representation of the form

$$\omega_k(z) = \int_0^\sigma E_{1/2}(-\tau^2 z; \mu)\, \tau^{\mu-1} \varphi_k(\tau)\, d\tau \qquad (k = 1, 2, \ldots), \tag{3.4}$$

where $\mu = \frac{3}{2} + \omega$ and the functions $\varphi_k(t) \in L_2(0, \sigma)$ $(k = 1, 2)$ are uniquely determined.

2. If $\nu \in (1 + \omega, 2 + \omega) \cap [0, 2)$ then the system of functions (3.3) lies in $W^{2,\omega}_{1/2,\sigma}$ and Theorem II (Sect. 0) again establishes an integral representation of the form

$$\omega_k^*(z) = \int_0^\sigma E_{1/2}(-\tau^2 z; \mu)\, \tau^{\mu-1} \varphi_k^*(\tau)\, d\tau \qquad (k = 0, 1, 2, \ldots), \tag{3.5}$$

where $\mu = \frac{3}{2} + \omega$ and the functions $\varphi_k^*(\tau) \in L_2(0, \sigma)$ are uniquely determined.

Using Theorem II (Sect. 0) we may invert formulae (3.4) and (3.5) and write the required functions $\varphi_k(\tau)$ and $\varphi_k^*(\tau)$ as improper integrals; but the calculation of them is, probably, rather complicated. However, one is able to find the explicit values of these integrals using the methods of [2].

Now we prove

Lemma 3.2. 1. *If* $0 \leqslant \nu < 2$, $0 < \mu < 3 + \nu$, *then the functions of the system* (3.2) *may be represented in the form* (3.4), *where*

$$\varphi_k(\tau) = -\frac{2\lambda_k \sigma^{-\nu}}{\mathscr{E}_\sigma(\lambda_k; \nu - 1)} E_{1/2}(-\lambda_k(\sigma - \tau)^2; \tilde{\mu})(\sigma - \tau)^{\tilde{\mu} - 1} \tag{3.6}$$

$(k = 1, 2, 3, \ldots)$ *and* $\tilde{\mu} = 3 + \nu - \mu$.

2. *If* $0 \leqslant \nu < 2$, $0 < \mu < 1 + \nu$, *then the functions of the system* (3.3) *may be represented in the form* (3.5), *where*

$$\begin{aligned} \varphi_0^*(\tau) &= \sigma^{-\nu} \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\tilde{\mu}^*)} (\sigma - \tau)^{\tilde{\mu}^* - 1}, \\ \varphi_k^*(\tau) &= \frac{2\sigma^{-\nu}}{\mathscr{E}_\sigma(\lambda_k; \nu - 1)} E_{1/2}(-\lambda_k(\sigma - \tau)^2; \tilde{\mu}^*)(\sigma - \tau)^{\tilde{\mu}^* - 1} \end{aligned} \tag{3.7}$$

$(k = 1, 2, \ldots)$ *and* $\mu^* = 1 + \nu - \mu$.

Proof. 1. If we substitute $\alpha=\mu$, $\beta=\tilde{\mu}=3+\nu-\mu$ (i.e., $\alpha+\beta-2=1+\nu$) and $\lambda=\lambda_k$, where λ_k is a zero of $E_{1/2}(-\sigma^2 z; 1+\nu)$, in the identity (3.1) of Lemma 3.1, then we obtain the identities

$$\int_0^\sigma E_{1/2}(-\tau^2 z;\mu)\,\tau^{\mu-1}E_{1/2}(-\lambda_k(\sigma-\tau)^2;\tilde{\mu})(\sigma-\tau)^{\tilde{\mu}-1}\,d\tau$$

$$=-\frac{E_{1/2}(-\sigma^2 z;1+\nu)}{z-\lambda_k}\,\sigma^{\nu}\qquad (k=1,2,...).$$

Now formula (3.6) of the lemma follows from the definition (3.2) of the system $\{\omega_k(z)\}_1^\infty$.

2. In order to establish formula (3.7) we first set $\lambda=0$ and then set $\lambda=\lambda_k$ in the identity (3.1′). Further, using the definition (3.3) of the system $\{\omega_k^*(z)\}_0^\infty$ we obtain formula (3.7).

Let us define the two following functional systems on the interval $(0,\sigma)$:

$$\{E_{1/2}(-\lambda_k t^2;\mu)\,t^{\mu-1}\}_1^\infty=\{e_\mu(t;\lambda_k)\}_1^\infty, \tag{3.8}$$

$$\{E_{1/2}(-\lambda_k t^2;\mu)\,t^{\mu-1}\}_0^\infty\equiv\{e_\mu(t;\lambda_k)\}_0^\infty, \tag{3.9}$$

where $\lambda_0=0$ by definition.

It is easy to see that $e_\mu(t;\lambda_0)=t^{\mu-1}/\Gamma(\mu)$, and the functions of both systems (38)–(3.9) belong to $L_1(0,\sigma)$ for $\mu>0$ and belong to $L_2(0,\sigma)$ for $\mu>\frac{1}{2}$.

Noting that all functions of the systems above are real valued, we prove

THEOREM 3.1. *The systems* (3.6) *and* (3.8), *like* (3.7) *and* (3.9), *are biorthogonal on* $(0,\sigma)$. *In other words,*

$$\int_0^\sigma \begin{matrix} e_\mu(\tau;\lambda_n)\,\varphi_k(\tau) \\ e_\mu(\tau;\lambda_n)\,\varphi_k^*(\tau)\end{matrix}\,d\tau=\begin{cases}\delta_{n,k};\ n,k=1,2,...,\\ \delta_{n,k};\ n,k=0,1,2,....\end{cases} \tag{3.10}$$

Proof. Let us set $z=\lambda_n$ in the integral representation (3.4) of $\omega_k(z)$. Since formula (1.26) of Lemma 1.1 easily implies $\omega_k(\lambda_n)=\delta_{k,n}(k,n\geqslant 1)$, we obtain the first part of (3.10). In order to establish the second part of (3.10), we set first $k=0$ in the representation (3.5) and note that by (3.3),

$$\omega_0^*(\lambda_0)=\omega_0^*(0)=1,\qquad \omega_0^*(\lambda_n)=0\qquad (n=1,2,...).$$

Therefore we obtain the second part of the formula (3.10), but for $k=0$, $n=0,1,2$, only. Finally, for $k\geqslant 1$ we have $\omega_k^*(\lambda_n)=\delta_{k,n}$ $(k,n=1,2,...)$ by (3.3) and using the representation (3.5) again we complete the proof.

3.2. Now we pass to proving expansion theorems with respect to the biorthogonal systems introduced above,

$$\{e_\mu(\tau; \lambda(k))\}_1^\infty, \qquad \{\varphi_k(\tau)\}_1^\infty \tag{3.11}$$

and

$$\{e_\mu(\tau; \lambda_k)\}_0^\infty, \qquad \{\varphi_k^*(\tau)\}_0^\infty. \tag{3.12}$$

THEOREM 3.2. *Let $\{\lambda_k\}_1^\infty \subset (0, +\infty)$ be the sequence of the zeros of*

$$\mathscr{E}_\sigma(z; \nu) = E_{1/2}(-\sigma^2 z; 1+\nu) \qquad (0 \leqslant \nu < 2). \tag{3.13}$$

Further, let

$$\tfrac{1}{2} < \mu < \tfrac{5}{2}, \qquad -\tfrac{3}{2} < \nu - \mu < -\tfrac{1}{2}. \tag{3.14}$$

Then

1. *In $L_2(0, \sigma)$ metric the series of the form*

$$\varphi(\tau) = \sum_{k=1}^{\infty} a_k \varphi_k(\tau) \tag{3.15}$$

establish a one to one continuous mapping of the space $l^{2,2(\mu-1)}$ of sequences $\{a_k\}_1^\infty$ with finite norm

$$\|\{a_k\}_1^\infty\|_{2,2(\mu-1)} = \left(\sum_{k=1}^{\infty} (1+k)^{2(\mu-1)} |a_k|^2\right)^{1/2} < +\infty \tag{3.16}$$

onto the space of functions $\varphi(\tau) \in L_2(0, \sigma)$. Here the coefficients of the series (3.15) *are determined via the formula*

$$a_k = \int_0^\sigma \varphi(t)\, e_\mu(t; \lambda_k)\, dt \qquad (k = 1, 2, ...) \tag{3.17}$$

and we have the two-sided estimate

$$\|\varphi\|_{L_2(0,\sigma)} \asymp \|\{a_k\}_1^\infty\|_{2,2(\mu-1)}. \tag{3.18}$$

2. *In $L_2(0, \sigma)$ metric the series of the form*

$$\varphi(\tau) = \sum_{k=1}^{\infty} b_k e_\mu(\tau; \lambda_k) \tag{3.19}$$

establish the same map of the space $l^{2,2(1-\mu)}$ of the sequences $\{b_k\}_1^\infty$ onto the spaces of all functions $\varphi(\tau) \in L_2(0, \sigma)$. Further, the formula

$$b_k = \int_0^\sigma \varphi(t)\, \varphi_k(t)\, dt \qquad (k = 1, 2, ...) \tag{3.20}$$

and the two-sided estimate

$$\|\varphi\|_{L_2(0,\sigma)} \asymp \|\{b_k\}_1^\infty\|_{2,2(1-\mu)} \tag{3.21}$$

hold true.

Proof. 1. By Theorem 2.1, if $p=2$ and

$$\nu \in (\omega, 1+\omega) \cap [0, 2), \qquad -1 < \omega < 1, \tag{3.22}$$

then the series of the form

$$\Phi(z) = \sum_{k=1}^\infty \bar{a}_k \omega_k(z) \tag{3.23}$$

establishes a one to one continuous mapping of $l^{2,2(\mu-1)}$ onto $W^{2,\omega}_{1/2,\sigma}$, where $\{\omega_k(z)\}_1^\infty$ is the system (3.2) of entire functions in $W^{2,\omega}_{1/2,\sigma}$ and $\mu = \omega + \frac{3}{2}$. Hence the conditions (3.22) and (3.14) are equivalent.

Further, Theorem 2.1 and the interpolation properties (3.2′) of $\{\omega_k(z)\}_1^\infty$ imply

$$\|\Phi\|^*_{2,\omega} \asymp \|\{a_k\}_1^\infty\|_{2,1+2\omega}, \qquad \Phi(\lambda_k) = \bar{a}_k \qquad (k = 1, 2,...). \tag{3.24}$$

On the other hand, in view of Theorems II and III (see Sect. 0) the formula

$$\Phi(z) = \int_0^\sigma E_{1/2}(-z\tau^2; \mu)\, \tau^{\mu-1} \overline{\varphi(\varepsilon)}\, d\tau \tag{3.25}$$

gives a one to one continuous mapping of $W^{2,\omega}_{1/2,\sigma}$ onto $L_2(0, \sigma)$, and

$$\|\Phi\|^*_{2,\omega} \asymp \|\varphi\|_{L_2(0,\sigma)}, \qquad \mu = \omega + \tfrac{3}{2}. \tag{3.26}$$

Now the two-sided inequality (3.18) of the theorem follows from (3.24) and (3.26) because $1 + 2\omega = 2(\mu - 1)$. Since the spaces $l^{2,2(\mu-1)}$ and $L_2(0, \sigma)$ are therefore homeomorphic, the proof of (1) will be completed if one notes that (3.24) and (3.25) imply

$$\begin{aligned}\Phi(\lambda_k) &= \bar{a}_k \\ &= \int_0^\sigma E_{1/2}(-\lambda_k\tau^2; \mu)\, \tau^{\mu-1} \overline{\varphi(\tau)}\, d\tau \\ &= \int_0^\sigma e_\mu(\tau; \lambda_k)\, \overline{\varphi(\tau)}\, d\tau \qquad (k = 1, 2,...).\end{aligned}$$

2. Now we consider the systems

$$\{\tilde{e}_\mu(\tau; \lambda_k)\}_1^\infty, \qquad \{\tilde{\varphi}_k(\tau)\}_1^\infty, \tag{3.11′}$$

putting

$$\tilde{\varphi}_k(\tau) = (1+k)^{1-\mu}\, \varphi_k(\tau), \qquad \tilde{e}_\mu(\tau; \lambda_k) = (1+k)^{\mu-1}\, e_\mu(\tau; \lambda_k) \tag{3.27}$$

$(k = 1, 2, \ldots)$. It is obvious that the systems (3.11′), as (3.11), are also biorthogonal on $(0, \sigma)$. Further, the above proved assertion (1) of the theorem will take the following form. The series of the form

$$\varphi(\tau) = \sum_{k=1}^{\infty} \tilde{a}_k \tilde{\varphi}_k(\tau) \tag{3.15′}$$

gives the same mapping of $L_2(0, \sigma)$ onto the space l^2 of sequences $\{\tilde{a}_k\}_1^\infty$ with norm

$$\|\{\tilde{a}_k\}_1^\infty\|_2 = \left(\sum_{k=1}^{\infty} |\tilde{a}_k|^2\right)^{1/2} < +\infty.$$

Further, instead of (3.17) and (3.18) we have

$$\tilde{a}_k = \int_0^\sigma \varphi(t)\, \tilde{e}_\mu(t; \lambda_k)\, dt \qquad (k \geqslant 1),$$

$$\|\varphi\|_{L_2(0,\sigma)} \asymp \|\{\tilde{a}_k\}_1^\infty\|_2.$$

Therefore, the system $\{\tilde{\varphi}_k(\tau)\}_1^\infty$ is a Riesz basis for $L_2(0, \sigma)$. Hence by a well-known theorem (see, for example [7, Chap. 6, Sect. 2]) the biorthogonal system to this is also a Riesz basis. Using this fact and returning to the system $\{\tilde{e}_\mu(\tau; \lambda_k)\}_1^\infty$ we complete the proof of 2.

THEOREM 3.3. *All the assertions of Theorem* 3.2 *are also valid for the biorthogonal systems* (3.12)

$$\{e_\mu(\tau; \lambda_k)\}_0^\infty, \qquad \{\varphi_k^*(\tau)\}_0^\infty,$$

if the conditions (3.14) *are replaced by*

$$\tfrac{1}{2} < \mu < \tfrac{5}{2}, \qquad -\tfrac{1}{2} < \nu - \mu < \tfrac{1}{2}. \tag{3.14′}$$

Proof. It is a copy of the proof of Theorem 3.2 above. We must only replace the use of Theorem 2.1 by the use of Theorem 2.2, where the condition (2.18) for $p = 2$ takes the form $\nu \in (1+\omega, 2+\omega) \cap [0, 2)$ $(-1 < \omega < 1)$. Finally, we again have $\mu = \omega + \frac{3}{2}$ and hence the conditions (3.14′) establish the validity of the theorem.

In conclusion we give some especially interesting particular cases of the expansion theorems.

THEOREM 3.4. *Let* $\{\lambda_k\}_1^\infty \subset (0, +\infty)$ *be the sequence of the zeros of* $E_{1/2}(-\sigma^2 z; 1+\nu)$, *where* $\nu \in (\frac{1}{2}, \frac{3}{2})$. *Then, after suitable normalizations, the biorthogonal systems*

$$\left\{\frac{\sin\sqrt{\lambda_k}\,t}{\sqrt{\lambda_k}}\right\}_1^\infty,$$
$$\left\{-\frac{2\sigma^{-\nu}\lambda_k(\sigma-t)^\nu}{E_{1/2}(-\sigma^2\lambda_k;\nu)}E_{1/2}(-\lambda_k(\sigma-t)^2;1+\nu)\right\}_1^\infty \tag{3.28}$$

and

$$\{\cos\sqrt{\lambda_k}\,t\}_0^\infty,$$
$$\left\{\nu\sigma^{-\nu}(\sigma-t)^{\nu-1}, \left\{\frac{2\sigma^{-\nu}(\sigma-t)^{\nu-1}}{E_{1/2}(-\sigma^2\lambda_k;\nu)}E_{1/2}(-\lambda_k(\sigma-t)^2;\nu\right\}_1^\infty\right\} \tag{3.29}$$

form Riesz bases for $L_2(0,\sigma)$.

Proof. Let us see that for $\mu = 2$ (resp. $\mu = 1$) we have

$$E_{1/2}(z;2) = \frac{\operatorname{sh}\sqrt{z}}{\sqrt{z}} \qquad (\text{resp. } E_{1/2}(z;1) = \operatorname{ch}\sqrt{z}).$$

Further, if $\nu \in (\frac{1}{2}, \frac{3}{2})$, then for $\mu = 2$ (resp. $\mu = 1$) the condition (3.14) (resp. (3.14')) of Theorem 3.2 (resp. 3.3) is satisfied. Hence the theorem follows from Theorem 3.2 (resp. 3.3) using the definitions (3.6), (3.7), (3.8), (3.9) of the biorthogonal systems (3.11) and (3.12).

Finally, let us see that for $\nu = 1$ the zeros of

$$E_{1/2}(-\sigma^2 z;2) = \frac{\sin\sigma\sqrt{z}}{\sigma\sqrt{z}} \qquad \text{are} \qquad \lambda_k = \left(\frac{\pi}{\sigma}k\right)^2 \ (k = 1, 2, ...).$$

In this case, of course, after simple renormalization, the systems (3.28) and (3.29) become the well-known systems which are orthonormal and closed in $L_2(0,\sigma)$

$$\left\{\sqrt{\frac{2}{\sigma}}\sin\frac{\pi k}{\sigma}t\right\}_1^\infty, \qquad \left\{\frac{1}{\sqrt{\sigma}}, \left\{\sqrt{\frac{2}{\sigma}}\cos\frac{\pi k}{\sigma}t\right\}_1^\infty\right\}. \tag{3.30}$$

Finally, we give the following theorem which is a corollary of the expansion Theorems 3.2 and 3.3.

THEOREM 3.5. *Let* $\{\lambda_k\}_1^\infty$ *be the zeros* of $E_{1/2}(-\delta^2 z; 1+\nu)$:

1. $\nu \in (\omega, 1+\omega) \cap [0, 2)$, $-1 < \omega < 1$; *then, after suitable normalization, the functional system*

$$\{E_{1/2}(-\lambda_k t^2;\mu)\}_1^\infty, \qquad \mu = \omega + \tfrac{3}{2},$$

forms a Riesz basis for the weighted functional space

$$L_2^{1+2\omega}(0, \sigma): \int_0^\sigma |\varphi(t)|^2 t^{1+2\omega} \, dt < +\infty.$$

2. *If* $v \in (\omega + 1, \omega + 2) \cap [0, 2)$, *then, after suitable normalization, the functional system*

$$\{E_{1/2}(-\lambda_k t^2; \mu)\}_0^\infty \qquad (\lambda_0 = 0)$$

also forms a Riesz basis for $L_2^{1+2\omega}(0, \sigma)$.

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